SCIENTIA – http://www.scientiajournal.org
International Review of Scientific Synthesis – ISSN 2282-2119
Vol. 125 – doi:10.12969/Scientia.Vol125.Sect2.Art03 - December 10th, 2013

Demonstration of Riemann Hypothesis

Diego Marin [1]

Abstract

We define an infinite summation which is proportional to the reverse of Riemann Zeta function $\zeta(s)$. Then we demonstrate that such function can have singularities only for $Re\, s = 1/n$ with $n \in \mathbb{N} \setminus 0$. Finally, using the functional equation, we reduce these possibilities to the only $Re\, s = 1/2$.

Keywords: Riemann Hypothesis; prime numbers

Contents

1 Target

Riemann hypothesis, proposed by Bernhard Riemann in the 1859, is a conjecture that regards an apparently simple function of complex variable s. Such function, called *Riemann Zeta Function*, is defined for $Re\, s > 1$ via the following summation

$$\zeta(s) = \sum_{n=1}^{+\infty} n^{-s}$$

For every integer n, an unique decomposition as product of (powers of) prime numbers exists. In this way

$$\zeta(s) = \prod_{p=2}^{+\infty} \sum_{j=0}^{+\infty} p^{-sj} = \prod_{p=2}^{+\infty} \frac{1}{1 - p^{-s}}$$

where we have used the summation rule for the geometric series

$$\sum_{j=1}^{+\infty} w^j = \frac{1}{1-w}$$

for $|w| < 1$. *Riemann Zeta function* can't have zeros in the convergence area, because no term in the product can be equal to zero. Nevertheless an holomorphic extension of $\zeta(s)$ can be defined over the entire complex plane $\mathbb{C}$, with the exception of $s = 1$. Such extension has infinite zeros corresponding to all negative even integers; that is, $\zeta(s) = 0$ when s is one of $-2, -4, -6, \ldots$. These are the so-called "trivial zeros".

Again for the holomorphic extension, for $s \neq 1$ we can prove the functional equation[2]

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

Leaving out the zeros of $\sin\left(\frac{\pi s}{2}\right)$ which aren't poles of $\Gamma(1-s)$, i.e. for $s = -2n$, $n \in \mathbb{N}$, any other zero s_0 must have a "mate" zero $s_0' = 1 - s_0$. Because there are no zeros for $Re\, s > 1$, functional equation implies that there are no zeros also for $Re\, s < 0$ (except for $s = -2n$). Other works have excluded also the presence of zeros for $Re\, s = 0$ and $Re\, s = 1$. As consequence, all the non trivial zeros of $\zeta(s)$ stay in the "critical strip" $0 < Re\, s < 1$.

Riemann hypothesis conjectures that all the non-trivial zeros have real part equal

[2]The first proof was given by Bernhard Riemann in the 10-page paper *Ueber die Anzahl der Primzahlen unter einer gegebenen Grsse* (usual English translation: *On the Number of Primes Less Than a Given Magnitude*) published in the November 1859 edition by Monatsberichte der Kniglich Preuischen Akademie der Wissenschaften zu Berlin.

to $\frac{1}{2}$. This is what we aim to demonstrate in the following.

2 Strategy

We start from the definition of $\zeta(s)$ as an infinite product of terms, one term for every prime number p, running from 2 to $+\infty$:

$$\zeta(s) = \prod_{p=2}^{+\infty} \frac{1}{1 - p^{-s}} \tag{1}$$

The product converges for $Re\, s > 1$. From here we fix $s = x + iy$ with $s \in \mathbb{C}$ and $x, y \in \mathbb{R}$. So the convergence condition is $x > 1$. Out of convergence area, $\zeta(s)$ is defined via holomorphic extension.

The derivative of a single term in the product (1) gives

$$\frac{d}{ds}\frac{1}{1 - p^{-s}} = -\frac{1}{(1 - p^{-s})^2}(\log p\; p^{-s}) = -\frac{1}{1 - p^{-s}} \cdot \frac{\log p}{p^s - 1}$$

Hence the derivative $\zeta'(s)$ of $\zeta(s)$ results

$$\zeta'(s) = \frac{d}{ds}\zeta(s) = -\prod_{p=2}^{+\infty} \frac{1}{1 - p^{-s}} \cdot \sum_{p'=2}^{+\infty} \frac{\log p'}{p'^s - 1}$$

and then

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{p=2}^{+\infty} \frac{\log p}{p^s - 1} \tag{2}$$

For briefness we set $C(s) = \frac{\zeta'(s)}{\zeta(s)}$. The sum (2) converges for $x > 1$. Otherwise we define $C(s)$ as the holomorphic extension of (2), recognizing it by the label "*H.e.*":

$$C(s) = -H.e. \sum_{n=1}^{\infty} \frac{\log p_n}{p_n^s - 1}$$

The uniqueness of the holomorphic extension ensures that $C(s) = \frac{\zeta'(s)}{\zeta(s)}$ also for $x \leq 1$. At this point we rely upon the following evidence:

Any singularity of $C(s)$ corresponds to a zero of $\zeta(s)$ and/or to a singularity of $\zeta'(s)$.

We can exclude that a zero of $\zeta'(s)$ hides a zero of $\zeta(s)$: in fact, for any holomorphic function $f(s)$, if a point s_0 exists which is a zero both for $f(s)$ and $f'(s)$, we have surely

$$\frac{f(s_0)}{f'(s_0)} = \lim_{s\to s_0} \frac{\sum_{k=1}^{\infty} f_k(s-s_0)^k}{\sum_{k=1}^{\infty} k f_k(s-s_0)^{k-1}} = \lim_{s\to s_0} \frac{f_{\hat{k}}(s-s_0)^{\hat{k}}}{\hat{k} f_{\hat{k}}(s-s_0)^{\hat{k}-1}} = \lim_{s\to s_0} \frac{s-s_0}{\hat{k}} = 0$$

where $\hat{k}$ is the minor k for which a Taylor coefficient f_k is $\neq 0$. Hence the zeros of $\zeta(s)$ are singularities for $C(s)$ also when they are zeros of both $\zeta(s)$ and $\zeta'(s)$.

For this reason we'll proceed with finding an ensemble of points which includes all the singularities of $C(s)$ and so, among them, all the zeros of $\zeta(s)$.

3 Application of the Euler-MacLaurin formula

We move from prime to integer numbers:

$$C(s) = -\sum_{n=1}^{\infty} \frac{\log p_n}{p_n^s - 1} \tag{3}$$

where p_n is the n-th prime number. For $s \neq 0$, no partial sum

$$C_1^M(s) = -\sum_{n=1}^{M} \frac{\log p_n}{p_n^s - 1}$$

with $M < +\infty$ has singularities. Hence, any function

$$C_M(s) = -\sum_{n=M}^{+\infty} \frac{\log p_n}{p_n^s - 1}$$

has the same singularities of $C(s)$ for $s \neq 0$. This permit us to work with $C_M(s)$ in place of $C(s)$, in such a way to exploit the freedom in choosing M. Consider now the Euler-MacLaurin theorem at leading order:

$$\left| \sum_{l=M}^{N} f(l,s) - \int_M^N f(w,s)dw \right| \leq F^N(s) \in \mathbb{R}^+$$

$$F^N(s) = \int_M^N dw \left| \frac{d}{dw} f(w,s) \right|$$

for some $f(w,a)$ analytic in $w \in \mathbb{R}$ and holomorphic in $s \in \mathbb{C}$ except at most for isolated points, provided that in such points we have $w \notin \mathbb{N}$.

Consider now $\lim_{N\to\infty} F^N(s) \stackrel{!}{=} F(s)$. If $F(s)$ exists (is finite) in a open set $x > A$ (with A some real number) except at most for isolated points, then the *Dominated Convergence Theorem* ensures (for $x > A$) that

$$\lim_{N\to\infty} \left| \sum_{l=M}^{N} f(l,s) - \int_{M}^{N} f(w,s)dw \right| \leq F(s) \tag{4}$$

Now it is possible that the sum and the integral in the left side converge only for $x > B$ with $B > A$. For $A < x \leq B$ we can separate the holomorphic extension of the summation ($H.e.$) from its divergent piece ($D.p.$):

$$\sum_{l=M}^{\infty} f(l,s) = H.e. \sum_{l=M}^{\infty} f(l,s) + D.p. \sum_{l=M}^{\infty} f(l,s)$$

We can do the same for the integral:

$$\int_{M}^{\infty} f(w,s)dw = H.e. \int_{M}^{\infty} f(w,s)dw + D.p. \int_{M}^{\infty} f(w,s)dw$$

Inserting them in (4) we obtain:

$$\left| H.e. \sum_{l=M}^{\infty} f(l,s) + D.p. \sum_{l=M}^{\infty} f(l,s) - H.e. \int_{M}^{\infty} f(w,s)dw - D.p. \int_{M}^{\infty} f(w,s)dw \right| \leq F(s)$$

Being $F(s)$ finite (except for isolated points), it has to be true

$$D.p. \sum_{l=M}^{\infty} f(l,s) = D.p. \int_{M}^{\infty} f(w,s)dw$$

and so

$$\left| H.e. \sum_{l=M}^{\infty} f(l,s) - H.e. \int_{M}^{\infty} f(w,s)dw \right| \leq F(s)$$

for $A < x \leq B$. Hence we can apply the Euler-Maclaurin formula not only for a comparison between sum and integral, but also between their holomorphic extensions, at least until $F(s)$ is finite. Our case is

$$\left| \sum_{n=M}^{\infty} \frac{\log p_n}{p_n^s - 1} - \int_{M}^{\infty} dt\, \frac{\log p(t)}{p(t)^s - 1} \right| \leq F(s)$$

Here $p(t)$ is any analytic function which posses analytic inverse $t(p)$ and satisfies $p(n) =$

p_n. Moreover

$$
\begin{aligned}
|\sum - \int| &\leq \int_M^\infty dt \left| \frac{d}{dt} \frac{\log p(t)}{p(t)^s - 1} \right| \\
&\leq \int_M^\infty dt \left| \frac{dp(t)}{dt} \left[\frac{1}{p(t)(p(t)^s - 1)} - \frac{sp(t)^{s-1} \log p(t)}{(p(t)^s - 1)^2} \right] \right| \qquad (5)
\end{aligned}
$$

The right side converges for $x > 0$ with the exceptions of isolated points.

The function $t(p_n)$ returns the cardinality n of the prime number p_n. This is equivalent to return how many prime numbers exist that are less than or equal to p_n, i.e. $t(p) = \pi(p)$, where $\pi(p)$ is the prime-counting function. A holomorphic definition was given by Riesel, Edwards and Derbyshire[3]:

$$
\pi(p) = \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{p^s}{s} \log \zeta(s) ds
$$

$$
\pi'(p) = \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} p^{s-1} \log \zeta(s) ds
$$

For $p \to +\infty$ we have

$$
\begin{aligned}
\pi(p) &\sim \frac{p}{\log p} \\
\pi'(p) = \frac{d\pi(p)}{dp} &\sim \frac{1}{\log p} \left(1 - \frac{1}{\log p}\right) \sim \frac{1}{\log p}
\end{aligned}
$$

Before going any further, we prove that (the holomorphic extension of) the integral in $|\sum - \int|$,

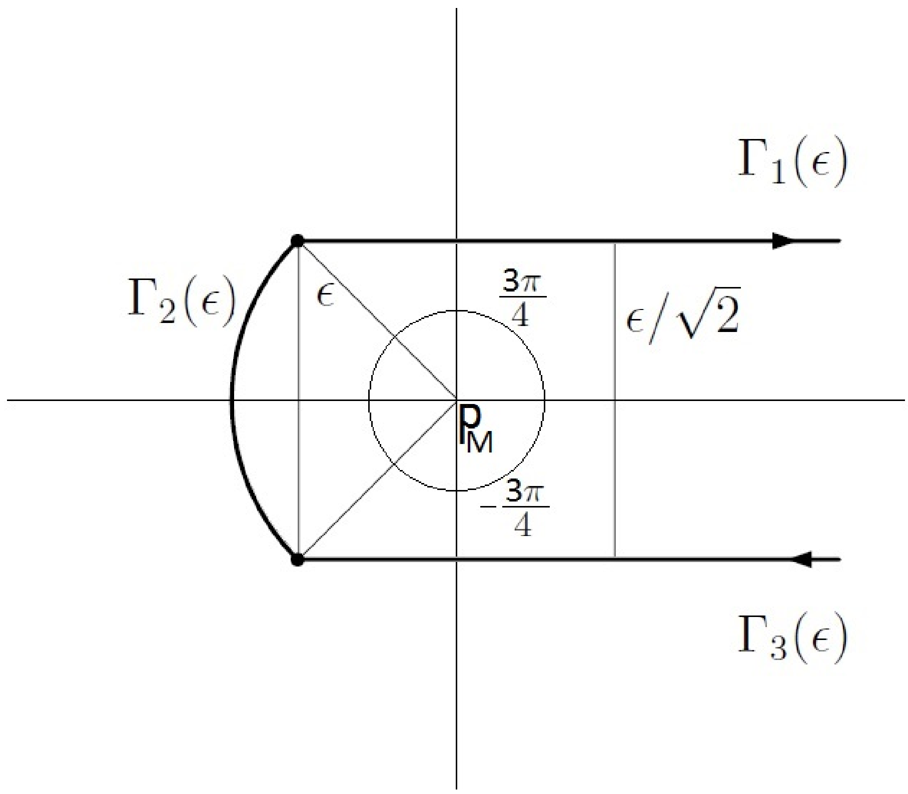
$$
H.e. \int_{p_M}^{+\infty} dp \, \frac{dt(p)}{dp} \, \frac{\log p}{p^s - 1} = H.e. \int_{p_M}^{+\infty} dp \, \pi'(p) \, \frac{\log p}{p^s - 1}, \qquad (6)
$$

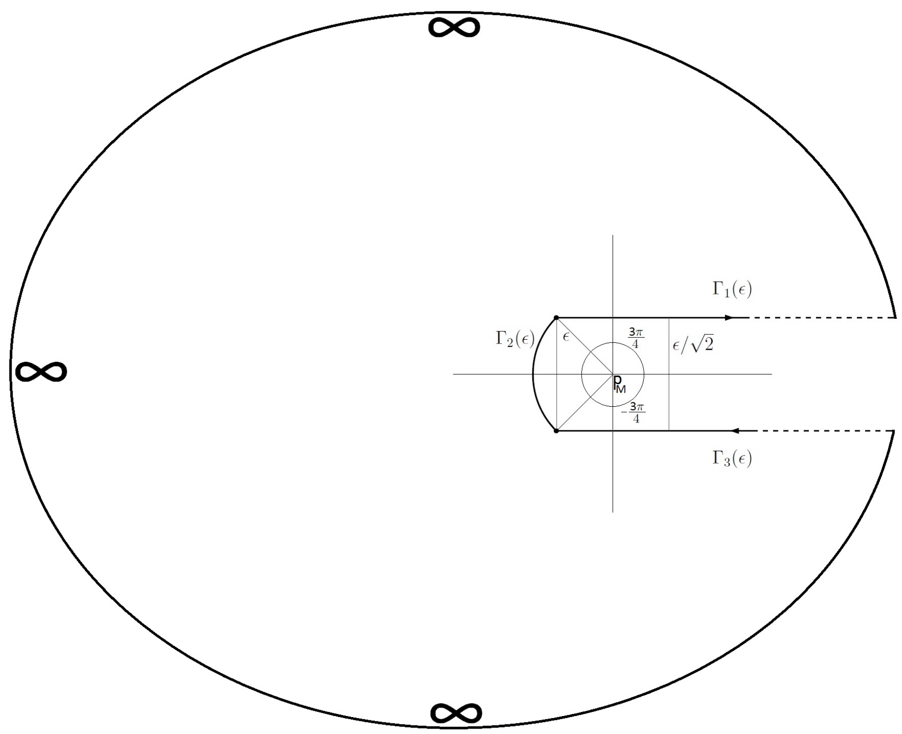
has no singularities. We introduce first a redefinition of the same integral which permits to calculate it also for $x \leq 1$. This is

$$
\frac{1}{2i \sin(\pi s)} \int_{C_\varepsilon} dp \, \pi'(p) \frac{\log p \, (-p)^{-s}}{1 - p^{-s}}, \qquad (7)
$$

which is equivalent to (6) for $x > 1$, so giving the relative holomorphic extension. The Cauchy principle ensures that integral maintains the same value for every choice of ε

[3]Riesel, H. “The Riemann Prime Number Formula” *Prime Numbers and Computer Methods for Factorization*, 2nd ed. Boston, MA: Birkhuser, pp. 50-52, 1994; Edwards, H. M. *Riemann's Zeta Function*. New York: Dover, 2001; Derbyshire, J. *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. New York: Penguin, 2004.





(until the path doesn't touch any pole of the integrated function); for demonstration we find useful to work in the limit $\varepsilon \to 0^+$.

The contribute at ∞ is null for $x > 1$. This is unique zone of interest to check the equivalence between (6) and (7); so, at this aim, we can forget about it.

$$\lim_{|p|\to+\infty} \pi'(p)\frac{\log p\ (-p)^{-s}}{1-p^{-s}} = 0 \qquad for \qquad x>1$$

Having said this, we proceed by calculating separately the contributes to the integral along Γ_1, Γ_3 and Γ_2:

$$\begin{aligned}\int_{\Gamma_1}[\quad] &= \lim_{\varepsilon\to 0^+}\int_{p_M}^{+\infty} dt\, \pi'(t+i\varepsilon)\frac{log(t+i\varepsilon)\, e^{-s\log(t+i\varepsilon)+i\pi s}}{1-e^{-s\log(t+i\varepsilon)}}\\ &= e^{i\pi s}\int_{p_M}^{+\infty} dp\, \pi'(p)\, \frac{\log p\, p^{-s}}{1-p^{-s}}\\ &= e^{i\pi s}\int_{p_M}^{+\infty} dp\, \pi'(p)\, \frac{\log p}{p^s-1}\end{aligned}$$

$$\begin{aligned}\int_{\Gamma_3}[\quad] &= \lim_{\varepsilon\to 0^+}\int_{+\infty}^{p_M} dt\, \pi'(t-i\varepsilon)\frac{log(t-i\varepsilon)\, e^{-s\log(t-i\varepsilon)-i\pi s}}{1-e^{-s\log(t-i\varepsilon)}}\\ &= -e^{-i\pi s}\int_{p_M}^{+\infty} dp\, \pi'(p)\, \frac{\log p\, p^{-s}}{1-p^{-s}}\\ &= -e^{-i\pi s}\int_{p_M}^{+\infty} dp\, \pi'(p)\, \frac{\log p}{p^s-1}\end{aligned}$$

For the integral in Γ_2, we pose $p=\varepsilon e^{i\theta}+p_M$, $dp = i\varepsilon e^{i\theta}d\theta$,

$$\begin{aligned}\int_{\Gamma_2}[\quad] &= \lim_{\varepsilon\to 0}\int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} d\theta\, \frac{\pi'(\varepsilon e^{i\theta}+p_M)\log(\varepsilon e^{i\theta}+p_M)(\varepsilon e^{i\theta}+p_M)^{-s}e^{i\pi s}i\varepsilon e^{i\theta}}{1-(\varepsilon e^{i\theta}+p_M)^{-s}}\\ &= \lim_{\varepsilon\to 0} i\varepsilon\int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} d\theta\, \frac{\pi'(p_M)\log(p_M)(p_M)^{-s}e^{i\pi s}e^{i\theta}}{1-(p_M)^{-s}}\\ &= \lim_{\varepsilon\to 0} i\varepsilon\, \frac{\pi'(p_M)\log(p_M)e^{i\pi s}}{p_M^s-1}\int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} d\theta\, e^{i\theta}\\ &= \lim_{\varepsilon\to 0} \varepsilon\, \frac{\pi'(p_M)\log(p_M)e^{i\pi s}}{p_M^s-1}\left[e^{i\frac{3\pi}{4}}-e^{-i\frac{3\pi}{4}}\right]\\ &= \lim_{\varepsilon\to 0} i\varepsilon\, \frac{\sqrt{2}\,\pi'(p_M)\log(p_M)e^{i\pi s}}{p_M^s-1} = 0\end{aligned}$$

In the end:

$$
\begin{aligned}
\frac{1}{2i\sin(\pi s)}\int_{C_\varepsilon} dp\,\pi'(p)\frac{\log p\;(-p)^{-s}}{1-p^{-s}} &= \frac{1}{2i\sin(\pi s)}\left[\int_{\Gamma_1}[\quad]+\int_{\Gamma_2}[\quad]+\int_{\Gamma_3}[\quad]\right]\\
&= \frac{1}{2i\sin(\pi s)}\left(e^{i\pi s}-e^{-i\pi s}\right)\int_{p_M}^{+\infty} dp\,\pi'(p)\,\frac{\log p}{p^s-1}\\
&= \frac{1}{2i\sin(\pi s)}2i\sin(\pi s)\int_{p_M}^{+\infty} dp\,\pi'(p)\,\frac{\log p}{p^s-1}\\
&= \int_{p_M}^{+\infty} dp\,\pi'(p)\,\frac{\log p}{p^s-1}
\end{aligned}
$$

CVD. At this stage we can doubtless affirm that (7) is the holomorphic extension of (6). Using the Cauchy's integral theorem, the integral in (7) can be transformed into a sum over the minimal circuitations around all the poles. These sit at $p = exp\left(\frac{2\pi ik}{s}\right)$, $k \in \mathbb{Z}$. A single term is the following:

$$
\sin(\pi s)\oint^{(k)}[\quad] = \lim_{\varepsilon\to 0^+}\frac{i\varepsilon}{2i}\int_0^{2\pi} e^{i\theta}d\theta\,\pi'(e^{\frac{2\pi ik}{s}}+\varepsilon e^{i\theta})\frac{\log(e^{\frac{2\pi ik}{s}}+\varepsilon e^{i\theta})}{(e^{\frac{2\pi ik}{s}}+\varepsilon e^{i\theta})^s-1}e^{i\pi s} =
$$

$$
= \lim_{\varepsilon\to 0^+}\frac{\varepsilon}{2}\int_0^{2\pi} e^{i\theta}d\theta\,\pi'(e^{\frac{2\pi ik}{s}}+\varepsilon e^{i\theta})\frac{\log(1+\varepsilon e^{i\theta-\frac{2\pi ik}{s}})+\frac{2\pi k}{s}}{e^{\frac{2\pi ik}{s}s}(1+\varepsilon e^{i\theta-\frac{2\pi ik}{s}})^s-1}e^{i\pi s} =
$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0^+}\frac{\varepsilon}{2}\int_0^{2\pi} e^{i\theta}d\theta\,\pi'(e^{\frac{2\pi ik}{s}}+\varepsilon e^{i\theta})\frac{\log(1+\varepsilon e^{i\theta-\frac{2\pi ik}{s}})+\frac{2\pi k}{s}}{(1+\varepsilon e^{i\theta-\frac{2\pi ik}{s}})^s-1}e^{i\pi s} &=\\
&= \lim_{\varepsilon\to 0^+}\frac{\varepsilon}{2}\int_0^{2\pi} e^{i\theta}d\theta\,\pi'(e^{\frac{2\pi ik}{s}}+\varepsilon e^{i\theta})\frac{\varepsilon e^{i\theta-\frac{2\pi ik}{s}}+\frac{2\pi k}{s}}{s\varepsilon e^{i\theta-\frac{2\pi ik}{s}}}e^{i\pi s} &=\\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2}\int_0^{2\pi} e^{i\theta}d\theta\,\pi'(e^{\frac{2\pi ik}{s}})\frac{\frac{2\pi k}{s}}{se^{i\theta-\frac{2\pi ik}{s}}}e^{i\pi s} &=\\
&= \frac{\pi k}{s^2}\pi'(e^{\frac{2\pi ik}{s}})e^{\frac{2\pi ik}{s}+i\pi s}\int_0^{2\pi} d\theta &=\\
&= \frac{2\pi^2 k}{s^2}\pi'(e^{\frac{2\pi ik}{s}})e^{\frac{2\pi ik}{s}+i\pi s}
\end{aligned}
$$

By summing over all the poles:

$$\sin(\pi s)\, H.e. \int_{p_M}^{+\infty} dp\, \pi'(p)\, \frac{\log p}{p^s - 1} = e^{i\pi s} \sum_{k=-\infty}^{+\infty} \frac{2\pi^2 k}{s^2} \pi'(e^{\frac{2\pi i k}{s}}) e^{\frac{2\pi k i}{s}}$$

$$= e^{i\pi s} \sum_{k=-\infty}^{+\infty} \frac{2\pi^2 k}{s^2} \pi'(e^{\frac{2\pi i k}{s}}) e^{\frac{2\pi i k s^*}{|s|^2}}$$

$$= \frac{\pi}{s^2} e^{i\pi s} \int_{-\infty}^{+\infty} \log\zeta(2+iw)dw \qquad \sum_{k=-\infty}^{+\infty} k e^{\frac{2\pi i k(1+iw)s^*}{|s|^2}} e^{\frac{2\pi i k s^*}{|s|^2}}$$

$$= \frac{\pi}{s^2} e^{i\pi s} \int_{-\infty}^{+\infty} \log\zeta(2+iw)dw \qquad \sum_{k=1}^{+\infty} k \left[e^{\frac{2\pi i k(2+iw)s^*}{|s|^2}} - e^{-\frac{2\pi i k(2+iw)s^*}{|s|^2}} \right]$$

Use now the summation rule

$$\sum_{k=1}^{+\infty} k e^{ka} = \frac{d}{da} \sum_{k=1}^{+\infty} e^{ka} = \frac{d}{da}\left[\frac{1}{1-e^a} - 1\right] = \frac{e^a}{(1-e^a)^2} = \frac{1}{(e^{-a/2} - e^{a/2})^2}$$

The last term in the right is the correct value of summation only for $Re\, a < 0$. Nevertheless, when we climb over the line $Re\, a = 0$, it gives the corresponding holomorphic extension. Hence we can use the rule without care of convergence criterium:

$$= \frac{\pi}{s^2} e^{i\pi s} \int_{-\infty}^{+\infty} \log\zeta(2+iw)dw \qquad \frac{1}{\left(e^{\frac{\pi i(2+iw)s^*}{|s|^2}} - e^{-\frac{\pi i(2+iw)s^*}{|s|^2}} \right)^2}$$

For $w \to +\infty$ the integrand goes like

$$\sim \frac{\pi}{s^2} e^{i\pi s} \log\zeta(2+iw)\, e^{-\frac{2\pi w s^*}{|s|^2}}$$

and so the integral converges at $+\infty$ (remember that $Re\, s^* = x > 0$). Similarly, for $w \to -\infty$ the integrand goes like

$$\sim \frac{\pi}{s^2} e^{i\pi s} \log\zeta(2+iw)\, e^{\frac{2\pi w s^*}{|s|^2}}$$

and so the integral converges at $-\infty$. Being know that $\zeta(2+iw)$ has neither zeros nor poles for $w \in \mathbb{R}$, we can say that (the *Holomorphic Extension* of) the integral in $|\sum - \int|$ has no singularities for $s \in \mathbb{C} \setminus \mathbb{R}$.

The inescapable conclusion is that all singularities of $C(s)$ in the critical strip emerge from the difference between it and the corresponding integral, i.e. they are among the isolated points where $F(s) = +\infty$.

4 Calculating the limiting function

Let recover the result (5):

$$\begin{aligned}
|\sum - \int| &\leq \int_M^{+\infty} dt \left|\frac{dp(t)}{dt}\right| \left|\left[\frac{1}{p(t)(p(t)^s-1)} - \frac{sp(t)^{s-1}\log p(t)}{(p(t)^s-1)^2}\right]\right| \\
&\leq \int_M^{+\infty} dt\, \frac{dp(t)}{dt} I_t \left|\left[\qquad\right]\right| - \int_M^{+\infty} dt\, \frac{dp(t)}{dt}[1-I_t] \left|\left[\qquad\right]\right|
\end{aligned}$$

where $I_t = 1$ if $\frac{dp(t)}{dt} \geq 0$ and $I_t = 0$ otherwise. Use now $dt\,\frac{dp(t)}{dt} = dp$ to achieve an advantageous change of variable:

$$\begin{aligned}
&\leq \int_{p_M}^{+\infty} dp\, I_{t(p)} \left|\left[\qquad\right]\right| - \int_{p_M}^{+\infty} dp\, [1-I_{t(p)}] \left|\left[\qquad\right]\right| \\
&\leq \int_{p_M}^{+\infty} dp\, I_{t(p)} \left|\left[\qquad\right]\right| + \int_{p_M}^{+\infty} dp\, [1-I_{t(p)}] \left|\left[\qquad\right]\right| \\
&\leq \int_{p_M}^{\infty} dp \left|\left[\frac{1}{p(p^s-1)} - \frac{sp^{s-1}\log p}{(p^s-1)^2}\right]\right| \\
&\leq \int_{p_M}^{\infty} dp \left|\frac{p^s - 1 - sp^{s-1}\log p}{p(p^s-1)^2}\right|
\end{aligned}$$

Now consider the following inequality:

$$\begin{aligned}
|p^s - 1|^2 &= (p^s-1)(p^{s^*}-1) = p^{s+s^*} - p^s - p^{s^*} + 1 \\
&= p^{2x} - 2p^x cos(y \log p) + 1 \\
&\geq p^{2x} - 2p^x + 1 = (p^x-1)^2
\end{aligned}$$

For $x > 0$ we have also

$$
\begin{aligned}
|p^s - 1 - sp^{s-1}\log p|^2 &= (p^s - 1 - sp^{s-1}\log p)(p^{s^*} - 1 - s^* p^{s^*-1}\log p) \\
&= p^{2x} - 2p^x cos(y\log p) + 1 - \\
&\quad -2xp^{2x-1}\log p + 2xp^{x-1}\log p\cos(y\log p) - \\
&\quad -2yp^{x-1}\log p\sin(y\log p) + |s|^2 p^{2x-2}\log^2 p \\
&\le p^{2x} + 2p^x + 1 + 2xp^{2x-1}\log p + 2xp^{x-1}\log p + \\
&\quad +2|y|p^{x-1}\log p + (x^2+y^2)p^{2x-2}\log^2 p
\end{aligned}
$$

$$
\begin{aligned}
&\le p^{2x} + 2p^x + 1 + 2xp^{2x-1}\log p + 2xp^{x-1}\log p + \\
&\quad +2|y|p^{x-1}\log p + (x^2+y^2)p^{2x-2}\log^2 p + \\
&\quad +2x|y|p^{2x-2}\log^2 p + 2|y|p^{2x-1}\log p = \\
&\qquad = (p^x + 1 + (x+|y|)p^{x-1}\log p)^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left|\sum - \int\right| &\le \int_{p_M}^{\infty} dp\, \frac{p^x + 1 + (x+|y|)p^{x-1}\log p}{p(p^x-1)^2} \\
&\le \frac{2}{x(1-p^x)} - \frac{\log(1-p^{-x})}{x} - \frac{(x+|y|)\log p((p^x-1)\Phi\big(p^x, 1, -\frac{1}{x}\big) + x)}{x^2 p(p^x-1)}\Bigg|_{p_M}^{\infty}
\end{aligned}
$$

$\Phi(w,a,b)$ is the *Lerch Transcendent*[4] which goes at $+\infty$ in the first variable as w^{-1}; in our case as p^{-x}. As consequence, the contribute at $+\infty$ is null. Finally

$$
\left|\sum - \int\right| \le \frac{2}{x(p_M^x-1)} + \frac{\log(1-p_M^{-x})}{x} + \frac{(x+|y|)\log p((p_M^x-1)\Phi\big(p_M^x, 1, -\frac{1}{x}\big) + x)}{x^2 p_M(p_M^x-1)}
$$

The *Lerch Transcendent* $\Phi(w,a,b)$ has singularities (if $a \in \mathbb{N}\setminus 0$) only for $b \in -\mathbb{N}\setminus 0$. In our case, $a = 1$ and so we have singularities for

$$
x = \frac{1}{n} \qquad with \qquad n \in \mathbb{N}\setminus 0
$$

[4] $\Phi(w,a,b)$ is usually defined as the holomorphic extension of $\Phi(w,a,b) = \frac{1}{\Gamma(w)}\int_0^{+\infty}\frac{t^{a-1}e^{-bt}}{1-we^{-t}}dt$ which works for $\{Re\, b > 0 \wedge Re\, a > 0 \wedge |w| < 1\} \cup \{Re\, b > 0 \wedge Re\, a > 1 \wedge |w| = 1\}$.

This means that the zeros of $\zeta(s)$ in the strip $0 < x < 1$ have to satisfy $x = \frac{1}{n}$ for some $n \in \mathbb{N} \setminus 0$. Moreover the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

reveals that if s_0 is a non-trivial zero, then $1 - s_0$ is a zero too. Hence we search for two integers m, n such that

$$\frac{1}{m} = 1 - \frac{1}{n},$$

but the unique solution of this equation is $m = n = 2$. Consequently, all the non-trivial zeros of zeta must have real part equal to 1/2. **CVD**